Discrete Mathematics 310 (2010) 2100–2112

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Characterizations of the convex geometries arising from the double shellings of posets

Kenji Kashiwabara *, Masataka Nakamura

Department of Systems Science, The University of Tokyo, Japan

ARTICLE INFO

Article history:
Received 25 August 2009
Received in revised form 25 March 2010
Accepted 29 March 2010
Available online 21 April 2010

Keywords:
Poset
Convex geometries
Trace operation
Double shelling

ABSTRACT

We investigate the class of double-shelling convex geometries. A double-shelling convex geometry is the collection of sets represented as the intersection of an ideal and a filter of a poset. The size of the stem of any rooted circuit of a double-shelling convex geometry is 2. We characterize the double-shelling convex geometries by the conditions that the rooted circuits should fulfill. Moreover we also characterize the same class in terms of trace-minimal forbidden minors.

1. Introduction

A convex geometry, introduced by Edelman and Jamison [4], is a combinatorial abstraction of convexity whereas a matroid is an abstraction of linear dependency. Convex geometries are derived from a variety of combinatorial objects, such as posets, affine point configurations, chordal graphs, semi-lattices, and so on. The complement of a convex geometry is known as an antimatroid.

A poset convex geometry is the collection of ideals of a poset. It is well known that the size of any stem of a convex geometry is 1 if and only if it is a poset convex geometry. On the other hand, the double-shelling convex geometry of a poset is the intersection of an ideal and a filter of a poset. Its shelling process consists of deleting a minimal or maximal element, which is the origin of the name 'double shelling'. For details, see [2].

In this paper, we characterize the class of double-shelling convex geometries.

The size of the stem of any rooted circuit of a double-shelling convex geometry is 2, which can be interpreted as the root of a rooted circuit being put between two 'exterior' elements. This is the first necessary condition (Condition 1 in Theorem 20).

The set of extreme elements of a double-shelling convex geometry consists of the minimal and the maximal elements of the poset. This leads to the second necessary condition that the extreme-element graph should be bipartite (Condition 2 in Theorem 20).

The third necessary condition can be described in three different ways. The first one is provided by the rules that the rooted circuits should satisfy (Condition 3 in Theorem 20), and the next one is described by the set of trace-minimal forbidden minors of size 5 or less (Condition 3′ in Theorem 39). The last one is the condition that the trace of the given convex geometry on any subset of size 5 is a double-shelling convex geometry (Condition 3″ in Corollary 40).

A necessary and sufficient condition for a convex geometry to be a double-shelling convex geometry consists of Conditions 1, 2, and one of Conditions 3 or 3′ and 3″ (Theorems 20 and 39 and Corollary 40). This is the main result of this paper. Note that Condition 3, Condition 3′ and Condition 3″ are all equivalent under Conditions 1 and 2.

* Corresponding author.
E-mail address: kashiwa@idea.c.u-tokyo.ac.jp (K. Kashiwabara).

0012-365X/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2010.03.031

The double-shelling convex geometries are also investigated by Semenova and Wehrung [10–12] from a lattice theoretic viewpoint. Chvátal [2] investigated the class of double-shelling convex geometries, too. He regarded a rooted circuit with a stem of size 2 as the ternary relation of betweenness and investigated the convex geometries with stems of size 2.

Although there are a large number of forbidden-minor type characterization theorems in matroid theory, there are only a few in the theory of convex geometries. Kashiwabara and Nakamura [5] characterize the class of edge-shelling convex geometries in terms of trace-minimal forbidden minors. Nakamura [7] gives a forbidden-minor characterization for the class of convex geometries of the graph search with respect to the operations of deletion and contraction. Okamoto and Nakamura [8] give that of the convex geometries of the graph line search.

2. Preliminaries

2.1. Closure systems

Let V be a finite ground set throughout this paper.

Since a convex geometry is a special case of closure systems, we begin with defining the closure system. The closure operator and the extreme operator are defined for a closure system. These operators play an important role for convex geometries.

Definition 1. A family $\mathcal{K}$ of subsets of V is a closure system if the following conditions hold.

1. $V \in \mathcal{K}$.
2. $A, B \in \mathcal{K}$ implies $A \cap B \in \mathcal{K}$.

An element of a closure system is called a *closed set*.

For a closure system, define $\tau(X) = \bigcap\{A \in \mathcal{K} \mid X \subseteq A\}$ for $X \subseteq V$. Since $\tau(X)$ is a closed set, it is the smallest set in all the closed sets including X. $\tau : 2^V \to 2^V$ is called the *closure operator* of the closure system.

Define $\mathrm{ex}(X) = \{x \in X \mid x \notin \tau(X - x)\}$ for $X \subseteq V$. An operator $\mathrm{ex} : 2^V \to 2^V$ is called the *extreme operator* of the closure system.

For an extreme operator, it is known that the following lemma holds.

Lemma 2 (*[1,9]*)**.** *For $A \subseteq B \subseteq V$, $\mathrm{ex}(B) \cap A \subseteq \mathrm{ex}(A)$ holds.*

For $e \in V$ and $X \subseteq V$, (X, e) with $e \notin X$ is called a *rooted set*. e is called the *root* of the rooted set, and X is called the *stem* of the rooted set. In the literature, for example [3,6], $(X \cup \{e\}, e)$ is sometimes called a rooted set. But when we write a rooted set (X, e), we adopt the notation satisfying that X does not contain e.

For a closure system $\mathcal{K}$ on V, a rooted set (X, e) is called a *rooted circuit* if X is a minimal set satisfying $e \in \tau(X)$.

2.2. Convex geometries

A convex geometry is a special case of a closure system.

Definition 3. For a closure system on V, it is a convex geometry if $\mathrm{ex}(X) = \mathrm{ex}(\tau(X))$ holds for any $X \subseteq V$.

The property $\mathrm{ex}(X) = \mathrm{ex}(\tau(X))$ is called the Klein–Milman property. An element of a convex geometry is called a *convex set*.

It is known that a convex geometry is also defined in terms of the axiom of rooted circuits.

Lemma 4 (*[3]*)**.** *A family $\mathcal{C}$ of rooted sets becomes the set of rooted circuits of a convex geometry if and only if the following two conditions hold.*

(1) *If (X, e) and (Y, e) belong to $\mathcal{C}$ with $X \subseteq Y$, then $X = Y$.*
(2) *If (X, e) and (Y, f) belong to $\mathcal{C}$ with $e \in Y$, then there exists $Z \subseteq X \cup Y - \{e\}$ with $(Z, f) \in \mathcal{C}$.*

For a convex geometry $\mathcal{K}$ on V, its trace on $T \subseteq V$ is $\{T \cap X \mid X \in \mathcal{K}\}$. Note that T may not be a convex set.

The following two lemmas follow from the definitions.

Lemma 5. *The trace of a convex geometry on any set is also a convex geometry. The rooted circuits of the convex geometry obtained as the trace of a convex geometry on T are $\{(X, e) \in \mathcal{C} \mid X \subseteq T, e \in T\}$, where $\mathcal{C}$ is composed of the rooted circuits of the original convex geometry.*

The next lemma gives the relation between the extreme operator and the rooted circuits.

Lemma 6. *For a closure system, $e \in \mathrm{ex}(X)$ if and only if there exists no rooted circuit (A, e) with $A \subseteq X$.*

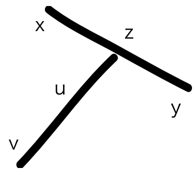

Fig. 1. Lemma 7.

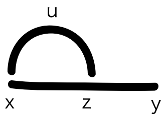

Fig. 2. Lemma 8.

Fig. 3. Lemma 9.

2.3. Convex geometries with stems of size 2

The class of double-shelling convex geometries is a subclass of convex geometries with stems of size 2. In this subsection, we consider convex geometries with stems of size 2. There are many important classes of convex geometries which belong to this class, e.g., the simplicial shelling of a chordal graph, the edge shelling of a tree, the vertex shelling of a tree, and so on.

The existence of a rooted circuit $(\{x, y\}, z)$ can be interpreted as z being between x and y.

We present some lemmas used in the proof of the main theorem. The following lemma follows from the case where the size of the stem is 2 in Lemma 4.

Lemma 7. *For any convex geometry with the stems of size* 2*, if* $(\{x, y\}, z) \in \mathcal{C}$ *and* $(\{v, z\}, u) \in \mathcal{C}$*, then* $(\{x, v\}, u) \in \mathcal{C}$*,* $(\{y, v\}, u) \in \mathcal{C}$*, or* $(\{x, y\}, u) \in \mathcal{C}$ *holds.*

Proof. By Lemma 4, there exists a rooted circuit $(X, u) \in \mathcal{C}$ such that $X \subseteq \{x, y, v\}$. Since the size of the stem is 2, $X = \{x, v\}, \{y, v\}$, or $\{x, y\}$. □

The following two lemmas are special cases of Lemma 7.

Lemma 8. *For any convex geometry with stems of size* 2*, if* $(\{x, y\}, z) \in \mathcal{C}$ *and* $(\{x, z\}, u) \in \mathcal{C}$*, then* $(\{x, y\}, u) \in \mathcal{C}$ *holds.*

Lemma 9. *For any convex geometry with stems of size* 2*, if* $(\{x, y\}, z) \in \mathcal{C}$ *and* $(\{u, z\}, y) \in \mathcal{C}$*,* $(\{x, u\}, y) \in \mathcal{C}$ *and* $(\{x, u\}, z) \in \mathcal{C}$ *hold.*

In this paper, we illustrate a rooted set as a line or a curve connecting corresponding three elements as in Figs. 1–3. Since these figures do not depict Hasse diagrams, the vertical position of elements in these figures has no meaning. When a figure depicts the Hasse diagram of a poset, we write explicitly 'the Hasse diagram' in its caption to avoid confusion.

2.4. Double-shelling convex geometries

For a poset, we obtain a convex geometry as follows.

Definition 10. For a poset P on V, $X \subseteq V$ is a convex set if $x, y \in X$ and $x \geq z \geq y$ imply $z \in X$. The collection of convex sets forms a convex geometry. We call such a convex geometry for some poset a double-shelling convex geometry.

A convex set of a double-shelling convex geometry is the intersection of a filter and an ideal.

The closure operator of the double-shelling convex geometry has the following property.

Lemma 11. *For a double-shelling convex geometry of a poset P, the closure operator τ for a set of size* 2 *is given as follows. When $x \geq y$, $\tau(\{x, y\}) = \{z \mid x \leq z \leq y\}$. When x and y are incomparable, $\tau(\{x, y\}) = \{x, y\}$.*

The next lemma shows the relation between a poset and the rooted circuits of the double-shelling convex geometry corresponding to the poset.

Lemma 12. *For the double-shelling convex geometry of a poset P, $(\{x, y\}, z)$ is a rooted circuit if and only if $x > z > y$ or $y > z > x$ holds.*

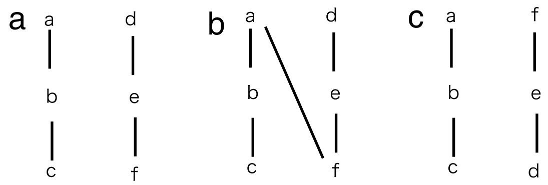

Fig. 4. Hasse diagrams of three posets inducing the same convex geometry.

Proof. When $(\{x, y\}, z)$ is a rooted circuit, $z \in \tau(\{x, y\})$. Therefore, by Lemma 11, $x > z > y$ or $y > z > x$ holds.

Conversely, suppose that $x > z > y$ or $y > z > x$. Then $z \in \tau(\{x, y\})$. Since $\{x, y\}$ is minimal in all the sets X such that $z \in \tau(X)$ and $x \notin X$, $(\{x, y\}, z)$ is a rooted circuit. □

Corollary 13. *For any double-shelling convex geometry, the stem of any rooted circuit has size* 2.

We call a chain consisting of three elements a *three-element chain*.

We define the connectivity of P as the connectivity of the Hasse diagram of P. Hence when $(\{x, y\}, z)$ is a rooted circuit, x, y, z belong to the same connected component.

For a double-shelling convex geometry, a poset giving rise to it is not necessarily unique. The next two lemmas indicate the redundancy of its representation. The next lemma follows immediately from Lemma 12.

Lemma 14. *For the double-shelling convex geometry of a poset P, we consider the poset P' by reversing the ordering on some connected component. Then P' and P give rise to the same double-shelling convex geometry.*

Lemma 15. *Consider the double-shelling convex geometry of a poset P. For a two-element maximal chain $\{x, y\}$, we consider the poset P' obtained by letting $\{x, y\}$ be incomparable. Then P' also induces the same double-shelling convex geometry. Conversely, for a maximal element y and a minimal element z, if y and z are incomparable, the poset obtained by letting them satisfy $y > z$ induces the same double-shelling convex geometry.*

Proof. By Lemma 12, a two-element maximal chain is irrelevant to the rooted circuits. Therefore they give rise to the same double-shelling convex geometry. □

Example 16. We present a poset and the double-shelling convex geometry corresponding to the poset. Consider the poset on $\{a, b, c, d, e, f\}$ shown in Fig. 4(a) as a Hasse diagram. This poset has two connected components and there exist two chains $a > b > c$ and $d > e > f$. Therefore the double-shelling convex geometry induced from this poset has two rooted circuits $(\{a, c\}, b)$ and $(\{d, f\}, e)$.

The poset of Fig. 4(b) has the same three-element chains, $a > b > c$ and $d > e > f$. Therefore this poset induces the same double-shelling convex geometry. This example supports Lemma 15.

The poset of Fig. 4(c) has three-element chains, $a > b > c$ and $f > e > d$. Therefore the double-shelling convex geometry induced from this poset has two rooted circuits $(\{a, c\}, b)$ and $(\{d, f\}, e)$. This is an example of Lemma 14. Therefore, the three posets give rise to the same double-shelling convex geometry.

We consider the trace operation for double-shelling convex geometries.

Lemma 17. *The class of double-shelling convex geometries is closed under the trace operation. The trace of the double-shelling convex geometry of a poset P on $X \subseteq P$ is a double-shelling convex geometry of the poset X.*

Proof. The proof of this lemma follows from Lemmas 5 and 12. □

By Lemma 5, the rooted circuits of the trace of the convex geometry on X are the collection of the rooted circuits which are included in X.

Lemma 18. *For a poset $Q \subseteq P$ and the double-shelling convex geometry of a poset P, $\mathrm{ex}(Q)$ consists of the maximal elements and the minimal elements of the poset Q.*

Proof. Assume that x is a minimal element and a maximal element of Q. Then there exists no pair y, z of elements such that $y < x < z$. Therefore there exists no rooted circuit whose root is x. By Lemma 6, $x \in \mathrm{ex}(X)$.

Conversely, assume that x is neither a maximal element nor a minimal element of Q. Then, there exist y and z such that $y > x > z$. Therefore $(\{y, z\}, x)$ is a rooted circuit. By Lemma 6, $x \notin \mathrm{ex}(X)$ holds. □

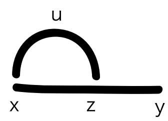

Fig. 5. Rule P.

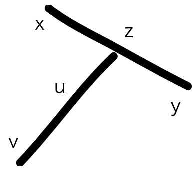

Fig. 6. Rule T.

3. The main theorem

3.1. Extreme-element graphs and the statement of the main theorem

For a convex geometry on V, we consider the following graph.

Let the vertex set be $\mathrm{ex}(V)$. For $x, y \in \mathrm{ex}(V)$, draw an edge $\{x, y\}$ when there exists a rooted circuit $(\{x, y\}, z)$ for some $z \in V$. We call such a graph the *extreme-element graph* of the convex geometry.

Lemma 19. *Consider a double-shelling convex geometry induced from poset P containing no maximal chain of at most two elements. Then $y, z \in \mathrm{ex}(V)$ belong to the same connected component of the extreme-element graph if and only if they belong to the same connected component of the Hasse diagram of P.*

Proof. Assume that y, z belong to the same connected component of the extreme-element graph. Then there exists a path between y and z in the extreme-element graph. Therefore we can make a path between y and z in the Hasse diagram of P.

Conversely, assume that $y, z \in \mathrm{ex}(V)$ belong to the same connected component of the Hasse diagram of P. Then there exists a path between y and z in the Hasse diagram. We make a subsequence $\langle u_0, \ldots, u_k\rangle$ from the sequence of nodes in the path satisfying the following two conditions.

(1) $u_0 = y$ and $u_k = z$,
(2) $(u_i > u_{i-1}$ and $u_i > u_{i+1})$ or $(u_i < u_{i-1}$ and $u_i < u_{i+1})$ for any $i = 1, \ldots, k-1$.

When $u_i > u_{i-1}$ and $u_i > u_{i+1}$, let u_i' be a maximal element of P with $u_i' > u_i$. When $u_i < u_{i-1}$ and $u_i < u_{i+1}$, let u_i' be a minimal element of P with $u_i' < u_i$. Let $u_0' = u_0$ and $u_k' = u_k$. By the assumption that P has no maximal chain of at most two elements, $\{u_i', u_{i+1}'\}$ is an edge in the extreme-element graph. Therefore we can make a path between y and z in the extreme-element graph. □

The next theorem is one of our main results.

Theorem 20. *For a convex geometry $\mathcal{K}$ with the rooted circuits $\mathcal{C}$, it is a double-shelling convex geometry if and only if the following conditions are satisfied.*

- *Condition* 1*: For any rooted circuit $(X, e) \in \mathcal{C}$ of $\mathcal{K}$, $|X| = 2$ holds.*
- *Condition* 2*: For any convex set $X \in \mathcal{K}$, the extreme-element graph of the trace of $\mathcal{K}$ on X is a bipartite graph.*
- *Condition* 3*: The following three rules are satisfied.*
  Rule P: $(\{x, y\}, z) \in \mathcal{C}$ and $(\{x, z\}, u) \in \mathcal{C}$ imply $(\{x, y\}, u) \in \mathcal{C}$ and $(\{u, y\}, z) \in \mathcal{C}$.
  Rule T: For $(\{x, y\}, z) \in \mathcal{C}$ and $(\{v, z\}, u) \in \mathcal{C}$, exactly one of the following two conditions holds.
  (1) $(\{x, v\}, u) \in \mathcal{C}$ *and* $(\{x, u\}, z) \in \mathcal{C}$ *and* $(\{x, v\}, z) \in \mathcal{C}$,
  (2) $(\{y, v\}, u) \in \mathcal{C}$ *and* $(\{y, u\}, z) \in \mathcal{C}$ *and* $(\{y, v\}, z) \in \mathcal{C}$.
  Rule X: For $(\{x, y\}, z) \in \mathcal{C}$ and $(\{v, u\}, z) \in \mathcal{C}$, exactly one of the following two conditions holds.
  (1) $(\{x, u\}, z) \in \mathcal{C}$ *and* $(\{y, v\}, z) \in \mathcal{C}$,
  (2) $(\{x, v\}, z) \in \mathcal{C}$ *and* $(\{y, u\}, z) \in \mathcal{C}$.
  Here, different letters indicate distinct elements (Figs. 5–7*).*

Note that, in Rule T and Rule X, when (1) does not hold, (2) must hold since exactly one of the two conditions (1) and (2) holds.

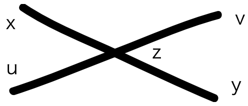

Fig. 7. Rule X.

3.2. Proof of the necessity of the main theorem

In this subsection, we prove the necessity of Theorem 20. That is, we show that any double-shelling convex geometry satisfies the three conditions in the theorem.

Condition 1 follows from Corollary 13.

Next we show Condition 2. By Lemmas 18 and 12, every edge in the extreme-element graph connects a minimal element and a maximal element. In other words, there exists no edge between maximal elements or between minimal elements. Hence it becomes a bipartite graph.

We show Condition 3 by showing the next three lemmas.

Lemma 21. *Any double-shelling convex geometry satisfies Rule P.*

Proof. Assume that $(\{x, y\}, z) \in \mathcal{C}$ and $(\{x, z\}, u) \in \mathcal{C}$. Then by Lemma 12, we have $x > z > y$ or $y > z > x$. Without loss of generality, we assume $x > z > y$. Similarly, we have $x > u > z$ or $z > u > x$. Since $x > z$, we have $x > u > z$. Therefore $x > u > z > y$. By Lemma 12 again, we have $(\{x, y\}, u) \in \mathcal{C}$ and $(\{u, y\}, z) \in \mathcal{C}$. □

Lemma 22. *Any double-shelling convex geometry satisfies Rule T.*

Proof. Consider the double-shelling convex geometry of a poset P. Consider x, y, z satisfying $x > z > y$. Then $z > u > v$ or $v > u > z$. We first consider the case of $z > u > v$. Then, by $x > z > u > v$, we have $x > u > v, x > z > u$, and $x > z > v$. Therefore $(\{x, v\}, u) \in \mathcal{C}$ and $(\{x, u\}, z) \in \mathcal{C}$ and $(\{x, v\}, z) \in \mathcal{C}$ hold. The case of $v > u > z$ is similarly shown. It is obvious that the two cases are exclusive. □

Lemma 23. *Any double-shelling convex geometry satisfies Rule X.*

Proof. Consider the double-shelling convex geometry of a poset P. Consider x, y, z satisfying $x > z > y$. In the case of $v > z > u$, we have $x > z > u$ and $v > z > y$. Therefore $(\{x, u\}, z) \in \mathcal{C}$ and $(\{y, v\}, z) \in \mathcal{C}$ hold. The case of $u > z > v$ is similarly shown. It is obvious that the two cases are exclusive. □

4. Proof of the sufficiency of the main theorem

We assume that a convex geometry with rooted circuits $\mathcal{C}$ satisfies Conditions 1, 2, and 3. We show that this convex geometry is a double-shelling convex geometry.

For that purpose, we construct a poset which induces the given convex geometry. We use mathematical induction on the size of the ground set V.

The case of size 1 is trivial. This makes the base case of the induction. Suppose $|V| \geq 2$.

We fix $x_0 \in \text{ex}(V)$ throughout the proof below. Recall that $\text{ex}(V)$ is non-empty by the definition of a convex geometry

Consider the trace of the given convex geometry on $V - \{x_0\}$. Note that the trace on $V - \{x_0\}$ also satisfies Conditions 1, 2, and 3. By the induction hypothesis, there exists a poset P on $V - \{x_0\}$ such that, for $y, u, z \in P$, $(\{y, u\}, z) \in \mathcal{C}$ if and only if $y >^P z >^P u$ or $u >^P z >^P y$, where $>^P$ is the strict partial order on P.

We assume that there exists no two-element maximal chain in P by Lemma 15.

It suffices to construct a poset P^* on V so that the given convex geometry is equal to the convex geometry induced from P^*.

We show the sufficiency using the following steps.

Step 1

Let Q be the set of elements which are contained in a rooted circuit containing x_0. We define a binary relation $>^Q$ on Q. We show that it is a strict partial order on Q.

Step 2

We define a poset $(P', >^{P'})$ by reversing the ordering on some components on P so that any minimal element of Q is a minimal element of P'.

Step 3

We define $>^*$ on V by combining $>^{P'}$ and $>^Q$. We show that it is a strict partial order on V.

Step 4

We check that strict partial order $>^*$ induces the given convex geometry. That is, any three-element chain in $(V, >^*)$ corresponds to a rooted circuit in the given convex geometry, and any rooted circuit corresponds to a three-element chain.

By proving the above steps, we complete the sufficiency proof of Theorem 20. We show an example to help the readers understand the sufficiency proof.

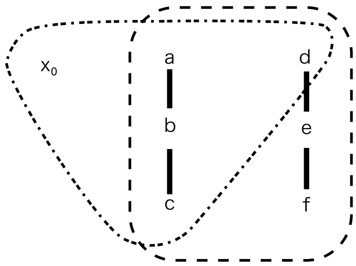

Fig. 8. Hasse diagram of P.

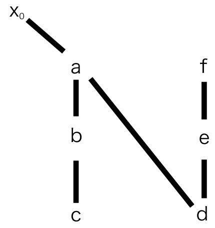

Fig. 9. Hasse diagram of the entire poset.

Example 24. Let P consist of six elements $\{a, b, c, d, e, f\}$. The order on P is defined by two three-element chains $a >^P b >^P c$ and $d >^P e >^P f$. Its rooted circuits are $(\{a, c\}, b)$ and $(\{d, f\}, e)$. Let V be the set of seven elements consisting of the above six elements and $\{x_0\}$. Suppose that the rooted circuits containing x_0 are $(\{x_0, d\}, a)$, $(\{x_0, b\}, a)$, $(\{x_0, c\}, a)$, $(\{x_0, c\}, b)$. This is a convex geometry and satisfies Conditions 1, 2, and 3. Q is $\{x_0, a, b, c, d\}$. The minimal elements of Q are c and d. Fig. 8 shows the orderings of P and Q.

Since c is a minimal element of P, and d is a maximal element of P, P' is defined by $a >^{P'} b >^{P'} c$ and $f >^{P'} e >^{P'} d$. Moreover $x_0 >^Q a >^Q b >^Q c$ and $a >^Q d$.

Therefore $x_0 >^* a >^* b >^* c$, and $a >^* d$, and $f >^* e >^* d$. We have the ordering on V shown in Fig. 9.

It can be easily checked that any three-element chain of the poset corresponds to a rooted circuit of the given convex geometry. Hence the given convex geometry is a double-shelling convex geometry.

4.1. Step 1

In this step, we introduce Q as the collection of the elements contained in some rooted circuits involving x_0. Then we show that Q is a poset. Recall that $x_0 \in \mathrm{ex}(V)$ is fixed. Recall that a poset P on $V - \{x_0\}$ satisfies that, for $y, u, z \in P$, $(\{y, u\}, z) \in \mathcal{C}$ if and only if $y >^P z >^P u$ or $u >^P z >^P y$.

We define a binary relation $>^Q$ on Q as follows.

Definition 25. Define $Q = \{x_0\} \cup \{y, z \in P \mid (\{x_0, z\}, y) \in \mathcal{C}\}$.

For $y \in Q - \{x_0\}$, we define $x_0 >^Q y$. For $y, z \in Q - \{x_0\}$, when $(\{x_0, z\}, y)$ is a rooted circuit, we define $y >^Q z$.

Lemma 26. *$>^Q$ is a strict partial order on Q.*

Proof. It suffices to show asymmetry and transitivity.

We first show asymmetry. Assume $y >^Q z$ and $z >^Q y$. Then we have $(\{x_0, z\}, y), (\{x_0, y\}, z) \in \mathcal{C}$. The trace on $\{x_0, y, z\}$ cannot be a convex geometry. Therefore asymmetry holds.

Next we consider the transitivity involving x_0. We show that $x_0 >^Q y$ and $y >^Q z$ imply $x_0 >^Q z$. This follows from the fact that $x_0 >^Q u$ for any element $u \in Q - \{x_0\}$ by definition.

Lastly, we consider the transitivity not involving x_0. Assume that $y >^Q z$ and $z >^Q u$. We may assume that y and u are different by asymmetry. Since $(\{x_0, z\}, y) \in \mathcal{C}$ and $(\{x_0, u\}, z) \in \mathcal{C}$, $(\{y, u\}, z) \in \mathcal{C}$ holds by Lemma 8. Therefore we have $u >^Q z$. $\square$

Note that asymmetry and transitivity follow only from any trace of a convex geometry being a convex geometry in the above proof. That is, the property of the double-shelling convex geometry is not used here.

4.2. Step 2

In this step, we construct P' from P so that it is compatible with the ordering of Q.

Lemma 27. *Every minimal element of Q with respect to $>^Q$ is a minimal or maximal element of P with respect to $>^P$.*

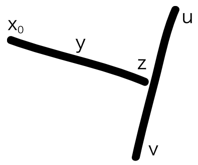

Fig. 10. Lemma 27.

Proof. Suppose that there exists a minimal element z of Q which is neither a minimal element nor a maximal element of P. There exists $y \in Q$ such that $(\{x_0, z\}, y) \in \mathcal{C}$. Since z is neither a minimal element nor a minimal element of P, there exist $u, v \in P$ such that $u >^P z >^P v$. Then $(\{u, v\}, z) \in \mathcal{C}$ by the definition of P. When y is neither u nor v, by Rule T (Fig. 10), $(\{x_0, u\}, z) \in \mathcal{C}$ or $(\{x_0, v\}, z) \in \mathcal{C}$ holds. Therefore $z >^Q u$ or $z >^Q v$, which contradicts z being a minimal element of Q. When y is equal to u, by Lemma 9 $(\{x_0, v\}, z)$ is a rooted circuit, a contradiction to z being a minimal element of Q. □

Corollary 28. *Every minimal element of Q with respect to $>^Q$ belongs to* $\mathrm{ex}(V)$.

Proof. Let y be such an element. Then there exists no rooted circuit in Q with root y since y is a minimal element of Q. Moreover there exists no rooted circuit in P with root y since y is a minimal or maximal element of P by Lemma 27. Since every rooted circuit is included in P or Q, there exists no rooted circuit with root y. $y \in \mathrm{ex}(V)$ holds by Lemma 6. □

Lemma 29. *Let y, z be minimal elements of Q. Moreover assume that y is a maximal element of P and z is a minimal element of P. Then y and z cannot belong to the same component of P.*

Proof. Suppose that y and z belong to the same component of P, from which we derive a contradiction.

By Corollary 28, $y, z \in \mathrm{ex}(V)$. That is, $x_0, y, z \in \mathrm{ex}(V)$ are vertices of the extreme-element graph.

Since y is a minimal element of Q, there exists $u \in Q$ such that $(\{x_0, y\}, u) \in \mathcal{C}$. Therefore $\{x_0, y\}$ is an edge of the extreme-element graph. Since z is a minimal element of Q, there exists $v \in Q$ such that $(\{x_0, z\}, v) \in \mathcal{C}$. Therefore $\{x_0, z\}$ is an edge of the extreme-element graph.

Since $\{x_0, y\}$ is an edge, x_0 and y belong to its different partite sets. Since $\{x_0, z\}$ is an edge, x_0 and z belong to different partite sets. When the extreme-element graph has the edge $\{y, z\}$, it is not a bipartite graph. Hence we can assume that $\{y, z\}$ is not an edge.

By Lemma 19, in the extreme-element graph on $\mathrm{ex}(V - \{x_0\})$, there exists a path $yu_0u_1 \cdots u_kz$ such that $u_0, \dots, u_k$ are minimal or maximal elements of P. k is odd since any edge connects a maximal element and a minimal element of P. Take z and y so that the path between z and y in the extreme-element graph is the shortest. Fix the shortest path between z and y. Let $Z = \tau(\{x_0, y, z, u_0, \dots, u_k\})$. Note that $\mathrm{ex}(Z) \subseteq \{x_0, y, z, u_0, \dots, u_k\}$ since $\mathrm{ex}(\tau(A)) \subseteq A$ holds generally. Therefore the shortest path is also shortest in the extreme-element graph on $\mathrm{ex}(Z - \{x_0\})$. Moreover, note that none of $u_0, \dots, u_k$ is a minimal element in Z with respect to $>^Q$ since otherwise we could make a shorter path using such u_i instead of y or z in the extreme-element graph on $\mathrm{ex}(Z - \{x_0\})$. By Lemma 2, $\mathrm{ex}(V) \cap Z \subseteq \mathrm{ex}(Z)$. Therefore $x_0, y, z \in \mathrm{ex}(Z)$. If all u_i belong to $\mathrm{ex}(Z)$, the extreme-element graph on $\mathrm{ex}(Z)$ of the trace of the convex geometry on Z has an odd cycle since $\{x_0, y\}$ and $\{x_0, z\}$ are edges in the extreme-element graph on $\mathrm{ex}(Z)$. Therefore it is not a bipartite as shown in Fig. 11, which contradicts Condition 2. Therefore it suffices to show that $\mathrm{ex}(Z) = \{x_0, y, z, u_0, \dots, u_k\}$.

Suppose that u_i is not an element of $\mathrm{ex}(Z)$ for some i. Since $u_i \in \mathrm{ex}(V - \{x_0\})$, there exists no rooted circuit (A, u_i) with $A \subseteq Z - \{x_0\}$. Therefore there exists a rooted circuit $(\{x_0, t\}, u_i)$ for some $t \in Z$. By choosing t to be a minimal element in Z with respect to $>^Q$, we have $t \in \mathrm{ex}(Z)$ by Corollary 28 applied to the trace of the convex geometry on Z. Therefore $t \in \{y, z, u_0, \dots, u_k\}$. By the assumption of the shortest path, the trace on Z does not have a minimal element of Q on $\mathrm{ex}(Z)$ except y, z. Therefore t should be y or z. That is, we have $(\{x_0, y\}, u_i) \in \mathcal{C}$ or $(\{x_0, z\}, u_i) \in \mathcal{C}$. We can consider the case of $(\{x_0, y\}, u_i) \in \mathcal{C}$ only without loss of generality.

Suppose that i is not k. Then $(\{u_i, u_{i+1}\}, w) \in \mathcal{C}$ for some $w \in Z$ shown in Fig. 12. By Rule T, either $(\{x_0, u_{i+1}\}, w) \in \mathcal{C}$ or $(\{y, u_{i+1}\}, w) \in \mathcal{C}$ holds. When $(\{x_0, u_{i+1}\}, w) \in \mathcal{C}$ holds, u_{i+1} is also a minimal element of Q, which contradicts the minimal elements of Q on Z being only y and z. When $(\{y, u_{i+1}\}, w) \in \mathcal{C}$ holds, we have a contradiction to the assumption of the shortest path. Hence we have $i = k$. Similarly we have $i = 0$. Therefore $k = 0$, a contradiction to k being odd. Therefore we have $\mathrm{ex}(Z) = \{x_0, y, z, u_0, \dots, u_k\}$. □

Definition 30. Define a poset P' on $V - \{x_0\}$ by reversing the ordering of some components of P so that any minimal element of Q with respect to $>^Q$ is a minimal element of P'.

By Lemmas 27 and 29, the above definition is well defined.

By Lemma 14, P and P' give rise to the same poset shelling convex geometry. Hence the following lemma holds.

Lemma 31. *For $y, z, u \in P'$, $(\{y, z\}, u) \in \mathcal{C}$ if and only if either $y >^{P'} u >^{P'} z$ or $z >^{P'} u >^{P'} y$ holds.*

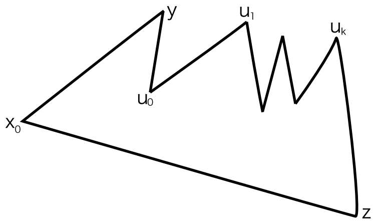

Fig. 11. Extreme-element graph on ex(Z).

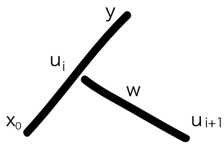

Fig. 12. Rule T.

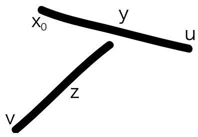

Fig. 13. Lemma 33 Case 1.

4.3. Step 3

In this step, we check that two posets Q and P' have a consistent intersection. Then we define a binary relation $>^*$ as the union of them, and we show that it is a strict partial order.

Lemma 32. *For $y, z \in P'$, $y >^{P'} z$ and $z >^{Q} y$ do not hold at the same time.*

Proof. Suppose that $y >^{P'} z$ and $z >^{Q} y$. When y is a minimal element of Q, it is also a minimal element of P', which contradicts $y >^{P'} z$. Therefore y is not a minimal element of Q. Therefore there exists a minimal element u of Q with $y >^{Q} u$. u is also a minimal element of P'. Since $z >^{Q} y$, $(\{x_0, y\}, z) \in \mathcal{C}$ holds. Since $y >^{Q} u$, $(\{x_0, u\}, y) \in \mathcal{C}$ holds. By Rule P, we have $(\{z, u\}, y) \in \mathcal{C}$. By Lemma 31, either $z >^{P'} y >^{P'} u$ or $u >^{P'} y >^{P'} z$ holds. Because of $y >^{P'} z$, $u >^{P'} y >^{P'} z$ holds, which contradicts u being a minimal element of P'. $\square$

Lemma 33. *If $y \in P' \cap Q$ and $z \in P'$, then $y >^{P'} z$ implies $y >^{Q} z$.*

Proof. Let $y \in Q$ and $y >^{P'} z$. It suffices to show $(\{x_0, z\}, y) \in \mathcal{C}$.

Since y is not a minimal element of P', it is not a minimal element of Q by Definition 30. Therefore, by taking a minimal element u of Q with $u <^{Q} y$, $(\{x_0, u\}, y) \in \mathcal{C}$ holds. By Definition 30, u is a minimal element of P'.

Moreover, there exists a three-element chain of P' containing y, z by the assumption that there exists no maximal two-element chain. We separate the proof into the three cases according to where v is inserted among y, z: (Case 1) $(\{y, v\}, z)$ is a rooted circuit, (Case 2) $(\{y, z\}, v)$ is a rooted circuit, and (Case 3) $(\{v, z\}, y)$ is a rooted circuit (Figs. 13–15).

Case 1: The case where $(\{y, v\}, z)$ is a rooted circuit.

Case 1-1: Assume that u is different from v and z.

By applying Rule T, exactly one of $(\{x_0, z\}, y) \in \mathcal{C}$ and $(\{u, y\}, z) \in \mathcal{C}$ holds.

When $(\{u, y\}, z) \in \mathcal{C}$, $y >^{P'} z$ implies $u >^{P'} y >^{P'} z$, which contradicts u being a minimal element of P'. Therefore we have $(\{x_0, z\}, y) \in \mathcal{C}$.

Case 1-2: When u is equal to v, $(\{x_0, z\}, y) \in \mathcal{C}$ can be proved by $(\{x_0, u\}, y) \in \mathcal{C}$.

Case 1-3: When u is equal to z, this case follows from Rule P.

Case 2: The case where $(\{y, z\}, v)$ is a rooted circuit. This case follows from an argument similar to that for Case 1.

Case 3: The case where $(\{v, z\}, y)$ is a rooted circuit.

Since u is a minimal element of P', it is not equal to v. When u and z are equal, $(\{x_0, z\}, y) \in \mathcal{C}$ holds. Therefore we can assume that u and z are different.

Since u is a minimal element of P' and $y >^{P'} z$, $(\{z, u\}, y) \in \mathcal{C}$ does not hold. By Rule X, we have $(\{x_0, z\}, y) \in \mathcal{C}$. $\square$

Therefore the set of elements of $P' \cap Q$ is an ideal of $>^{P'}$.

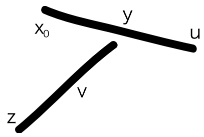

Fig. 14. Lemma 33 Case 2.

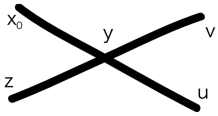

Fig. 15. Lemma 33 Case 3.

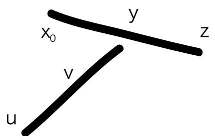

Fig. 16. Lemma 35 Case 1.

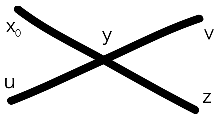

Fig. 17. Lemma 35 Case 2.

Lemma 34. *If* $(\{x_0, u\}, z) \in \mathcal{C}$ *for some* $z \in V$, u *is not a maximal element of* P'.

Proof. Suppose that u is a maximal element of P'. Then u is not a minimal element of P'. Therefore u is not a minimal element of Q with respect to $>^Q$ by the definition of P'. Then $(\{x_0, w\}, u) \in \mathcal{C}$ for some $w \in V$. By Rule P, $(\{z, w\}, u) \in \mathcal{C}$. Since $z, w, u \in P'$, $z >^{P'} u >^{P'} w$ or $w >^{P'} u >^{P'} z$ holds, which contradicts the assumption that u is a maximal element of P'. □

Lemma 35. *Assume that* y *is not a maximal element of* P'. *Then* $y >^Q z$ *implies* $y >^{P'} z$.

Proof. Since $y >^Q z$, $(\{x_0, z\}, y)$ is a rooted circuit. We separate the proof into two cases (Figs. 16 and 17).

Case 1: The case where $(\{y, u\}, v) \in \mathcal{C}$ is a rooted circuit with $u >^{P'} v >^{P'} y$. We may take u to be a maximal element of P'.

By Rule T, exactly one of $(\{u, x_0\}, y) \in \mathcal{C}$ and $(\{u, z\}, y) \in \mathcal{C}$ holds. By Lemma 34, $(\{u, x_0\}, y) \in \mathcal{C}$ contradicts u being a maximal element of P'. Therefore we have $(\{u, x_0\}, y) \notin \mathcal{C}$. Therefore $(\{u, z\}, y) \in \mathcal{C}$. Then $y >^{P'} z$.

Case 2: The case where $(\{v, u\}, y) \in \mathcal{C}$ is a rooted circuit with $u >^{P'} y >^{P'} v$.

With an argument similar to that for Case 1, $(\{u, x_0\}, y) \in \mathcal{C}$ cannot hold. By Rule X and $u >^{P'} y$, we have $(\{u, z\}, y) \in \mathcal{C}$ and $y >^{P'} z$. □

Definition 36. For $y, z \in V$, we define $y >^* z$ as $y >^{P'} z$ or $y >^Q z$.

Lemma 37. $>^*$ *is a strict partial order on* V.

Proof. Asymmetry follows from Lemma 32 and the fact that $>^{P'}$ and $>^Q$ are asymmetric.

We prove transitivity. Assume that $y >^* z >^* u$.

When $y >^{P'} z >^{P'} u$, $y >^* u$ follows from the transitivity of $>^{P'}$. On the other hand, when $y >^Q z >^Q u$, $y >^* u$ follows from the transitivity of $>^Q$ by Lemma 26.

Next we show that $y >^Q z$ and $z >^{P'} u$ imply $y >^* u$. By Lemma 33, $z >^{P'} u$ implies $z >^Q u$. Therefore $y >^Q u$ follows from the transitivity of $>^Q$.

Lastly, we show that $y >^{P'} z$ and $z >^Q u$ imply $y >^* u$. Since z is not a maximal element of P', $y >^* u$ follows from Lemma 35. □

4.4. Step 4

In this step, we show that the convex geometry arising from P^* is the same as the given convex geometry.

Lemma 38. *The convex geometry arising from P^* is the same as the given convex geometry. That is, for $y, z, u \in V$, $y >^* z >^* u$ or $u >^* z >^* y$ holds if and only if $(\{y, u\}, z) \in \mathcal{C}$ holds.*

Proof. Assume that a three-element chain with respect to $>^*$ is given. We separate the proof into two cases according to whether it contains x_0 or not.

First, we consider the case involving x_0. Assume that $x_0 >^* z >^* u$ since x_0 is a maximal element in $(V, >^*)$. Then we have $x_0 >^Q z >^* u$. By Lemma 33 and $z \in Q$, we have $z >^Q u$. By the definition of $>^Q$, we have $(\{x_0, u\}, z) \in \mathcal{C}$.

Next, we consider the case not involving x_0. For $y, z, u \in P'$, assume that $u >^* z >^* y$. When $u >^{P'} z >^{P'} y$, we have $(\{u, y\}, z) \in \mathcal{C}$ by Lemma 31. When $u >^Q z >^Q y$, we have $(\{x_0, z\}, u) \in \mathcal{C}$ and $(\{x_0, y\}, z) \in \mathcal{C}$. Then by Rule P, we have $(\{u, y\}, z) \in \mathcal{C}$.

When $u >^{P'} z >^Q y$, we have $u >^{P'} z >^{P'} y$ by Lemma 35. Therefore $(\{u, y\}, z) \in \mathcal{C}$ by Lemma 31. The case of $u >^Q z >^{P'} y$ results in the case of $u >^Q z >^Q y$ by Lemma 33.

We consider the converse direction by separating the proof into cases according to whether x_0 is involved or not.

First, we show that $(\{x_0, u\}, z) \in \mathcal{C}$ implies $x_0 >^* z >^* u$. By definition, this follows from $x_0 >^Q z >^Q u$.

Next, we show that, for $y, u, z \in P'$, $(\{y, u\}, z) \in \mathcal{C}$ implies $y >^* z >^* u$ or $u >^* z >^* y$. This follows from Lemma 31. □

By the above steps 1–4, we have completed the proof of Theorem 20.

5. Variants of the main theorem

In Theorem 20, Condition 3 is described in terms of the conditions that the rooted circuits should satisfy. We can restate Condition 3 in other forms.

Theorem 39. *A convex geometry $\mathcal{K}$ with the rooted circuits $\mathcal{C}$ is a double-shelling convex geometry if and only if the following conditions are satisfied.*

- *Condition 1: For any rooted circuit $(X, e) \in \mathcal{C}$ of $\mathcal{K}$, $|X| = 2$ holds.*
- *Condition 2: For any convex set $X \in \mathcal{K}$, the extreme-element graph of the trace of $\mathcal{K}$ on X is a bipartite graph.*
- *Condition 3: The convex geometry $\mathcal{K}$ contains none of the following convex geometries as a trace.*
  - *Type O: $(\{x, z\}, y), (\{x, z\}, u), (\{y, z\}, u)$.*
  - *Type B: $(\{x, z\}, y), (\{x, z\}, u), (\{y, z\}, u), (\{x, y\}, u)$.*
  - *Type X: $(\{x, y\}, z), (\{u, v\}, z)$.*
  - *Type K: $(\{x, y\}, z), (\{u, y\}, z), (\{u, v\}, z)$.*
  - *Type R: $(\{x, y\}, z), (\{x, v\}, z), (\{u, v\}, z), (\{x, y\}, u)$.*
  - *Type A: $(\{x, z\}, y), (\{x, v\}, u), (\{y, v\}, u)$.*
  - *Type U: $(\{x, y\}, z), (\{x, y\}, u), (\{x, y\}, v), (\{z, v\}, u)$.*
  - *Type Y: $(\{x, y\}, z), (\{x, y\}, u), (\{z, v\}, u)$.*

Proof. We first show that any double-shelling convex geometry satisfies Condition 3′. It can be seen that any convex geometry in Condition 3′ is not a double-shelling convex geometry as follows. Since Type O and Type B do not satisfy Rule P, they are not double-shelling convex geometries by Lemma 21. Since Type X and Type K and Type R do not satisfy Rule X, they are not double-shelling convex geometries by Lemma 23. Since Type A, Type U and Type Y do not satisfy Rule T, they are not double-shelling convex geometries by Lemma 22. Therefore, by Lemma 17, a convex geometry containing one of them as a trace is not a double-shelling convex geometry.

Conversely, we show that a convex geometry which satisfies Conditions 1, 2, and 3′ satisfies Condition 3 in Theorem 20. Suppose that Condition 3 does not hold. Then there exists a convex geometry of size at most 5 that is not a double-shelling convex geometry by Lemmas 21–23. By enumerating all convex geometries on five elements thoroughly, we can conclude that there exist four trace-minimal forbidden minors of the double-shelling convex geometries besides those listed in Condition 3′. These four trace-minimal forbidden minors are shown in Fig. 19. Even though a convex geometry satisfies Condition 3′, we see that it may contain one of the four trace-minimal forbidden minors by considering trace-minimal forbidden minors of size at most 5.

The extreme-element graphs of these four trace-minimal forbidden minors are not bipartite. We show that any convex geometry having such a minor as a trace does not satisfy Condition 2. Assume that a convex geometry has a forbidden minor in Fig. 19. Let $X \subseteq V$ be the ground set of the forbidden minor. Then the extreme-element graph of the trace on X is not bipartite. By Definition 3, $\mathrm{ex}(X) = \mathrm{ex}(\tau(X))$ holds. Therefore the extreme-element graph of closed set $\tau(X)$ contains the extreme-element graph on X as a subgraph. Therefore it does not satisfy Condition 2. Therefore the forbidden minors in Fig. 18 are sufficient for Condition 3′. □

The next statement is another form of the main theorem.

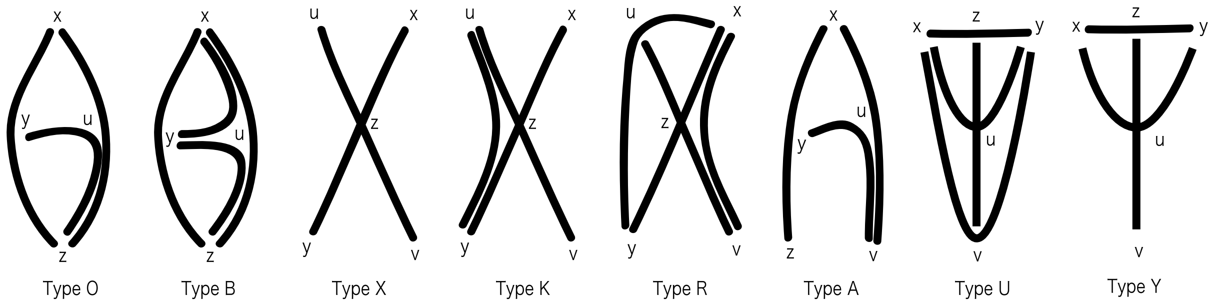

Fig. 18. Trace-minimal forbidden minors of double-shelling convex geometries.

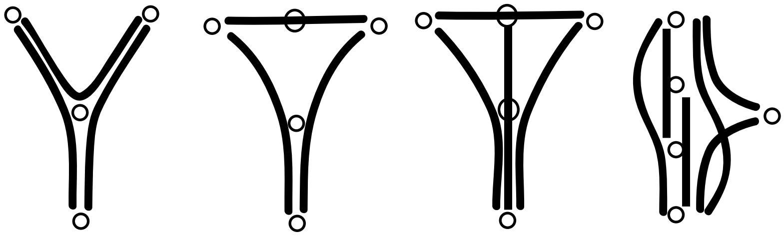

Fig. 19. The other trace-minimal forbidden minors on the 5-set.

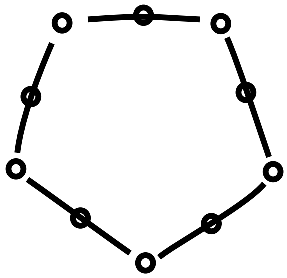

Fig. 20. A forbidden minor on ten elements.

Corollary 40. *For a convex geometry $\mathcal{K}$ with the rooted circuits $\mathcal{C}$, it is a double-shelling convex geometry if and only if the following conditions are satisfied.*

- *Condition* 1*: For any rooted circuit $(X, e) \in \mathcal{C}$ of $\mathcal{K}$, $|X| = 2$ holds.*
- *Condition* 2*: For any convex set $X \in \mathcal{K}$, the extreme-element graph of the trace of the convex geometry $\mathcal{K}$ on X is a bipartite graph.*
- *Condition* $3''$*: The trace on any five elements is a double-shelling convex geometry.*

Proof. It follows from Lemmas 21–23 that Condition $3''$ implies Condition 3.

Conversely, we show that Condition $3'$ together with Condition 2 implies Condition $3''$. Consider five arbitrary elements. Assume that the geometry satisfies Condition 2 and Condition $3'$. Like in the proof of Theorem 39, we have that it does not contain any trace-minimal forbidden minor of size at most 5. Therefore Condition $3''$ holds. □

The class of double-shelling convex geometries is closed under trace operation. Therefore theoretically, it may be characterized by trace-minimal forbidden minors even if we omit Condition 2. But this needs an infinite number of trace-minimal forbidden minors. For example, the convex geometry on the ten elements in Fig. 20 is a trace-minimal forbidden minor for double-shelling convex geometries. The extreme elements $\mathrm{ex}(V)$ of this convex geometry consist of five elements and its extreme-element graph is a 5-cycle, which is not bipartite. We can construct an infinite sequence of trace-minimal forbidden minors using other odd numbers instead of 5 in this example.

References

[1] K. Ando, Extreme point axioms for closure spaces, Discrete Mathematics 306 (24) (2006) 3181–3188.
[2] Vašek Chvátal, 'Antimatroids, Betweenness, Convexity' in Research Trends in Combinatorial Optimization Bonn 2008, Springer, Berlin, 2009, pp. 57–64.
[3] B.L. Dietrich, A circuit set characterization of antimatroids, Journal of Combinatorial Theory. Series B 43 (1987) 314–321.
[4] P.H. Edelman, R. Jamison, The theory of convex geometries, Geometriae Dedicata 19 (1985) 247–274.

[5] K. Kashiwabara, M. Nakamura, A characterization of the edge-shelling convex geometries of trees. arXiv:math.CO/0908.3456.

[6] B. Korte, L. Lovász, R. Schrader, Greedoids, Springer, Berlin, 1991.

[7] M. Nakamura, Excluded-minor characterizations of antimatroids arisen from posets and graph searches, Discrete Applied Mathematics 129 (2–3) (2003) 487–498.

[8] Y. Okamoto, M. Nakamura, The forbidden minor characterization of line-search antimatroids of rooted digraphs, Discrete Applied Mathematics 131 (2003) 523–533.

[9] J.L. Pfaltz, Closure lattices, Discrete Mathematics 154 (1996) 217–236.

[10] Marina Semenova, Friedrich Wehrung, Sublattices of lattices of order-convex sets, I. The main representation theorem, Journal of Algebra 277 (2) (2004) 825–860.

[11] Marina Semenova, Friedrich Wehrung, Sublattices of lattices of order-convex sets, II. Posets of finite length, International Journal of Algebra and Computation 13 (5) (2003) 543–564.

[12] Marina Semenova, Friedrich Wehrung, Sublattices of lattices of order-convex sets, III. The case of totally ordered sets, International Journal of Algebra and Computation 14 (3) (2004) 357–387.